

Partial regularity for elliptic systems with VMO-coefficients

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Abstract. We establish partial Hölder continuity for vector-valued solutions $u : \Omega \rightarrow \mathbb{R}^N$ to inhomogeneous elliptic systems of the type:

$$-\operatorname{div}(A(x, u, Du)) = f(x, u, Du) \quad \text{in } \Omega,$$

where the coefficients $A : \Omega \times \mathbb{R}^N \times \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^N) \rightarrow \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ are possibly discontinuous with respect to x . More precisely, we assume a VMO-condition with respect to the x and continuity with respect to u and prove Hölder continuity of the solutions outside of singular sets.

Keywords. Nonlinear elliptic systems, Partial regularity, VMO-coefficients, \mathcal{A} -harmonic approximation.

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1 Introduction

In this paper, we consider the second order nonlinear elliptic systems in divergence form of the following type:

$$-\operatorname{div}(A(x, u, Du)) = f(x, u, Du) \quad \text{in } \Omega. \quad (1.1)$$

Here Ω is bounded domain in \mathbb{R}^n , u takes values in \mathbb{R}^N with coefficients $A : \Omega \times \mathbb{R}^N \times \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^N) \rightarrow \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^N)$.

The aim of this paper is to obtain a partial regularity result of weak solutions to (1.1) with discontinuous coefficients. More precisely, we assume that the partial mapping $x \mapsto A(x, u, \xi)/(1+|\xi|)^{p-1}$ has vanishing mean oscillation (VMO), uniformly in (u, ξ) . This means that A satisfies an estimate

$$|A(x, u, \xi) - (A(\cdot, u, \xi))_{x_0, \rho}| \leq V_{x_0}(x, \rho)(1 + |\xi|)^{p-1},$$

where $V_{x_0} : \mathbb{R}^n \times [0, \rho_0] \rightarrow [0, 2L]$ are bounded functions with

$$\lim_{\rho \searrow 0} V(\rho) = 0, \quad V(\rho) := \sup_{x_0 \in \Omega} \sup_{0 < r \leq \rho} \int_{B_r(x_0) \cap \Omega} V_{x_0}(x, r) dx.$$

We also assume that $u \mapsto A(x, u, \xi)/(1+|\xi|)^{p-1}$ is continuous, that is, there exists a modulus of continuity $\omega : [0, \infty) \rightarrow [0, \infty)$ such that an estimate

$$|A(x, u, \xi) - A(x, u_0, \xi)| \leq L\omega(|u - u_0|^2)(1 + |\xi|)^{p-1}$$

holds.

Regularity results under a VMO-condition have been established by Zheng [11] for quasi-linear elliptic systems or integral functionals. General functionals with VMO-coefficients were considered by Ragusa and Tachikawa [10], who generalized the low-dimensional results from problems with continuous coefficients

to the case of VMO-coefficients. In particular, these results require that the dimension of domain is small, for example, $n \leq p + 2$ is required to obtain the Hölder continuity of the minimizers in [10]. In contrast, Bögelein, Duzaar, Habermann and Scheven [1] gives the regularity result for homogeneous nonlinear elliptic system without dimension conditions.

Stronger assumptions such as the Hölder continuity with respect to (x, u) or a Dini-type condition lead to partial C^1 -regularity with a quantitative modulus of continuity for Du ; the modulus of continuity can be determined in dependence on the modulus of continuity of the coefficients (cf. Giaquinta and Modica [7], Duzaar and Grotowski [5], Duzaar and Gastel [4], Chen and Tan [3], Qiu [9] and the references therein).

Our aim is to extend the homogeneous system result in [1] to inhomogeneous system. Therefore we assume the same structure conditions to coefficients A as in [1]. Under a suitable assumption to inhomogeneous term, we obtain Hölder continuity of weak solution (See Theorem 2.2).

Our proof is based on so-called \mathcal{A} -harmonic approximation (cf. [5, Lemma 2.1]; see also Lemma 3.2), introduced by Duzaar and Grotowski. They gives a simplified (direct) proof of regularity results to the systems with Hölder continuous coefficients and a natural growth condition, without L^p - L^2 -estimates for Du .

We close this section by briefly summarizing the notation used in this paper. As note above, we consider a bounded domain $\Omega \subset \mathbb{R}^n$, and maps from Ω to \mathbb{R}^N , where we take $n \geq 2$, $N \geq 1$. For a given set X we denote by $\mathcal{L}^n(X)$ as n -dimensional Lebesgue measure. We write $B_\rho(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < \rho\}$. For bounded set $X \subset \mathbb{R}^n$ with $\mathcal{L}^n(X) > 0$, we denote the average of a given function $g \in L^1(X, \mathbb{R}^N)$ by $\bar{f}_X g dx$, that is, $\bar{f}_X g dx = \frac{1}{\mathcal{L}^n(X)} \int_X g dx$. In particular, we write $g_{x_0, \rho} = \bar{f}_{B_\rho(x_0) \cap \Omega} g dx$. We write $\text{Bil}(\text{Hom}(\mathbb{R}^n, \mathbb{R}^N))$ for the space of bilinear forms on the space $\text{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ of linear maps from \mathbb{R}^n to \mathbb{R}^N .

2 Statement of the results

Definition 2.1. We say $u \in W^{1,p}(\Omega, \mathbb{R}^N)$, $p \geq 2$ is a weak solution of (1.1) if u satisfies

$$\int_{\Omega} \langle A(x, u, Du), D\varphi \rangle dx = \int_{\Omega} \langle f, \varphi \rangle dx \quad (2.1)$$

for all $\varphi \in C_0^\infty(\Omega, \mathbb{R}^N)$, where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product on \mathbb{R}^N or \mathbb{R}^{nN} .

We assume following structure conditions.

(H1) $A(x, u, \xi)$ is differentiable in ξ with bounded and continuous derivatives, that is, there exists $L \geq 1$ such that

$$|A(x, u, \xi)| + (1 + |\xi|) |D_\xi A(x, u, \xi)| \leq L(1 + |\xi|)^{p-1} \quad (2.2)$$

for all $x \in \Omega$, $u \in \mathbb{R}^N$ and $\xi \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^N)$. Moreover, this infers the modulus of continuity function $\mu : [0, \infty) \rightarrow [0, \infty)$ such that μ is bounded, concave, non-decreasing and we have

$$|D_\xi A(x, u, \xi) - D_\xi A(x, u, \xi_0)| \leq L\mu \left(\frac{|\xi - \xi_0|}{1 + |\xi| + |\xi_0|} \right) (1 + |\xi| + |\xi_0|)^{p-2} \quad (2.3)$$

for all $x \in \Omega$, $u \in \mathbb{R}^N$, $\xi, \xi_0 \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^N)$. Without loss of generality, we may assume $\mu \leq 1$.

(H2) $A(x, u, \xi)$ is uniformly strongly elliptic, that is, for some $\lambda > 0$ we have

$$\left\langle D_\xi A(x, u, \xi) \nu, \nu \right\rangle := \sum_{\substack{1 \leq i, k \leq N \\ 1 \leq j, l \leq n}} D_{\xi_{\beta}^j} A_{\alpha}^i(x, u, \xi) \nu_i^\alpha \nu_j^\beta \geq \lambda |\nu|^2 (1 + |\xi|)^{p-2} \quad (2.4)$$

for all $x \in \Omega$, $u \in \mathbb{R}^N$, $\xi, \nu \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^N)$.

(H3) $A(x, u, \xi)$ is continuous with u . More precisely, there exists bounded, concave and non-decreasing function $\omega : [0, \infty) \rightarrow [0, \infty)$ satisfying

$$|A(x, u, \xi) - A(x, u_0, \xi)| \leq L\omega(|u - u_0|^2)(1 + |\xi|)^{p-1} \quad (2.5)$$

for all $x \in \Omega$, $u, u_0 \in \mathbb{R}^N$, $\xi \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^N)$. Without loss of generality, we may assume $\omega \leq 1$.

(H4) $x \mapsto A(x, u, \xi)/(1 + |\xi|)^{p-1}$ fulfils the following VMO-conditions uniformly in u and ξ :

$$|A(x, u, \xi) - (A(\cdot, u, \xi))_{x_0, \rho}| \leq V_{x_0}(x, \rho)(1 + |\xi|)^{p-1}, \quad \text{for all } x \in B_\rho(x_0)$$

whenever $x_0 \in \Omega$, $0 < \rho < \rho_0$, $u \in \mathbb{R}^N$ and $\xi \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^N)$, where $\rho_0 > 0$ and $V_{x_0} : \mathbb{R}^n \times [0, \rho_0] \rightarrow [0, 2L]$ are bounded functions satisfying

$$\lim_{\rho \searrow 0} V(\rho) = 0, \quad V(\rho) := \sup_{x_0 \in \Omega} \sup_{0 < r \leq \rho} \int_{B_r(x_0) \cap \Omega} V_{x_0}(x, r) dx. \quad (2.6)$$

(H5) $f(x, u, \xi)$ has p -growth, that is, there exist constants $a, b \geq 0$, with a possibly depending on $M > 0$, such that

$$|f(x, u, \xi)| \leq a|\xi|^p + b \quad (2.7)$$

for all $x \in \Omega$, $u \in \mathbb{R}^N$ with $|u| \leq M$ and $\xi \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^N)$.

Now, we are ready to state our main theorem.

Theorem 2.2. *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$ be a bounded weak solution of (1.1) under the structure conditions (H1), (H2), (H3), (H4) and (H5) satisfying $\|u\|_\infty \leq M$ and $\lambda > 2^{(9p-10)/2}a(M)M$. Then there exists an open set $\Omega_u \subseteq \Omega$ with $\mathcal{L}^n(\Omega \setminus \Omega_u) = 0$ such that $u \in C_{\text{loc}}^{0,\alpha}(\Omega_u, \mathbb{R}^N)$ for every $\alpha \in (0, 1)$. Moreover, we have $\Omega \setminus \Omega_u \subseteq \Sigma_1 \cup \Sigma_2$, where*

$$\begin{aligned} \Sigma_1 &:= \left\{ x_0 \in \Omega : \liminf_{\rho \searrow 0} \int_{B_\rho(x_0)} |Du - (Du)_{x_0, \rho}|^p dx > 0 \right\}, \\ \Sigma_2 &:= \left\{ x_0 \in \Omega : \limsup_{\rho \searrow 0} |(Du)_{x_0, \rho}| = \infty \right\}. \end{aligned}$$

3 Preliminaries

In this section we present \mathcal{A} -harmonic approximation lemma and some standard estimates for the proof of the regularity theorem.

First we state the definition of \mathcal{A} -harmonic function and recall \mathcal{A} -harmonic approximation lemma as below.

Definition 3.1. *For a given $\mathcal{A} \in \text{Bil}(\text{Hom}(\mathbb{R}^n, \mathbb{R}^N))$, we say $h \in W^{1,p}(\Omega, \mathbb{R}^N)$ is \mathcal{A} -harmonic function, if h satisfies*

$$\int_{\Omega} \mathcal{A}(Dh, D\varphi) dx = 0$$

for all $\varphi \in C_0^\infty(\Omega, \mathbb{R}^N)$.

Lemma 3.2 ([1, Lemma 2.3]). *Let $\lambda > 0$, $L > 0$, $p \geq 2$ and $n, N \in \mathbb{N}$ with $n \geq 2$ given. For every $\varepsilon > 0$, there exists a constant $\delta = \delta(n, N, L, \lambda, \varepsilon) \in (0, 1]$ such that the following holds: assume that $\gamma \in [0, 1]$ and $\mathcal{A} \in \text{Bil}(\text{Hom}(\mathbb{R}^n, \mathbb{R}^N))$ with the property*

$$\mathcal{A}(\nu, \nu) \geq \lambda |\nu|^2, \quad \text{for all } \nu \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^N), \quad (3.1)$$

$$\mathcal{A}(\nu, \tilde{\nu}) \leq L |\nu| |\tilde{\nu}|, \quad \text{for all } \nu, \tilde{\nu} \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^N). \quad (3.2)$$

Furthermore, let $g \in W^{1,2}(B_\rho(x_0), \mathbb{R}^N)$ be an approximately \mathcal{A} -harmonic map in sense that there holds

$$\oint_{B_\rho(x_0)} \{|Dg|^2 + \gamma^{p-2} |Dg|^p\} dx \leq 1, \quad (3.3)$$

$$\left| \oint_{B_\rho(x_0)} \mathcal{A}(Dg, D\varphi) dx \right| \leq \delta \sup_{B_\rho(x_0)} |D\varphi|, \quad \text{for all } \varphi \in C_c^1(B_\rho(x_0), \mathbb{R}^N). \quad (3.4)$$

Then there exists an \mathcal{A} -harmonic function h that satisfies

$$\oint_{B_\rho(x_0)} \left\{ \left| \frac{h-g}{\rho} \right|^2 + \gamma^{p-2} \left| \frac{h-g}{\rho} \right|^p \right\} dx \leq \varepsilon, \quad (3.5)$$

$$\oint_{B_\rho(x_0)} \{|Dh|^2 + \gamma^{p-2} |Dh|^p\} dx \leq c(n, p). \quad (3.6)$$

Next is a standard estimates for the solutions to homogeneous second order elliptic systems with constant coefficients, due originally to Campanato [2, Teorema 9.2]. For convenience, we state the estimate in a slightly general form than original one.

Theorem 3.3 ([5, Theorem 2.3]). *Consider \mathcal{A} , λ and L as in Lemma 3.2. Then there exists $C_0 \geq 1$ depending only on n, N, λ and L such that any \mathcal{A} -harmonic function h on $B_{\rho/2}(x_0)$ satisfies*

$$\left(\frac{\rho}{2}\right)^2 \sup_{B_{\rho/4}(x_0)} |Dh|^2 + \left(\frac{\rho}{2}\right)^4 \sup_{B_{\rho/4}(x_0)} |D^2 h|^2 \leq C_0 \left(\frac{\rho}{2}\right)^2 \oint_{B_{\rho/2}(x_0)} |Dh|^2 dx. \quad (3.7)$$

We state the Poincaré inequality in a convenient form. The proof can be found in several literature, for example [6, Proposition 3.10].

Lemma 3.4. *There exists $C_P \geq 1$ depending only on n such that every $u \in W^{1,p}(B_\rho(x_0), \mathbb{R}^N)$ satisfies*

$$\int_{B_\rho(x_0)} |u - u_{x_0,\rho}|^p dx \leq C_P \rho^p \int_{B_\rho(x_0)} |Du|^p dx. \quad (3.8)$$

Given a function $u \in L^2(B_\rho(x_0), \mathbb{R}^N)$, where $x_0 \in \mathbb{R}^n$ and $\rho > 0$. We write $\ell_{x_0,\rho}$ is the minimizer of the functional

$$\ell \mapsto \oint_{B_\rho(x_0)} |u - \ell|^2 dx \quad (3.9)$$

among affine functions $\ell : \mathbb{R}^n \rightarrow \mathbb{R}^N$. Let write $\ell_{x_0,\rho}(x) := \ell_{x_0,\rho}(x_0) + D\ell_{x_0,\rho}(x - x_0)$. It is easy to check that $\ell_{x_0,\rho}(x_0) = u_{x_0,\rho}$ and

$$D\ell_{x_0,\rho} = \frac{n+2}{\rho^2} \oint_{B_\rho(x_0)} u \otimes (x - x_0) dx, \quad (3.10)$$

where $\xi \otimes \zeta = \xi_i \zeta^\alpha$. Based on this formula, elementary calculations yield the following estimates.

Lemma 3.5 ([1, Lemma 2.1]). Assume $u \in L^2(B_\rho(x_0), \mathbb{R}^N)$, $x_0 \in \mathbb{R}^n$, $\rho > 0$ and $0 < \theta \leq 1$. With $\ell_{x_0, \rho}$ and $\ell_{x_0, \theta\rho}$, we denote the affine functions from \mathbb{R}^n to \mathbb{R}^N defined as above for the radii ρ and $\theta\rho$ respectively. Then we have

$$|D\ell_{x_0, \rho} - D\ell_{x_0, \theta\rho}|^2 \leq \frac{n(n+2)}{(\theta\rho)^2} \int_{B_{\theta\rho}(x_0)} |u - \ell_{x_0, \rho}|^2 dx, \quad (3.11)$$

and more generally,

$$|D\ell_{x_0, \rho} - D\ell|^2 \leq \frac{n(n+2)}{\rho^2} \int_{B_\rho(x_0)} |u - \ell|^2 dx, \quad (3.12)$$

for all affine functions $\ell : \mathbb{R}^n \rightarrow \mathbb{R}^N$.

The estimate (3.12) implies, in particular, that $\ell_{x_0, \rho}$ has the following quasi-minimizing property for the L^p -norm.

Lemma 3.6. Consider the minimizer of (3.9), that is, $\ell_{x_0, \rho}$. For any affine functions $\ell : \mathbb{R}^n \rightarrow \mathbb{R}^N$ and $p \geq 2$ we have

$$\int_{B_\rho(x_0)} |u - \ell_{x_0, \rho}|^p dx \leq \tilde{c}(n, p) \int_{B_\rho(x_0)} |u - \ell|^p dx. \quad (3.13)$$

Proof. We write $\ell(x)$ as $\ell(x_0) + D\ell(x - x_0)$. First, we have

$$\begin{aligned} & \int_{B_\rho(x_0)} |u - \ell_{x_0, \rho}|^p dx \\ & \leq 3^{p-1} \left[\int_{B_\rho(x_0)} |u - \ell|^p dx + |u_{x_0, \rho} - \ell(x_0)|^p + \rho^p |D\ell_{x_0, \rho} - D\ell|^p \right]. \end{aligned} \quad (3.14)$$

Note that $\int_{B_\rho(x_0)} D\ell(x - x_0) dx = 0$ holds. Hence we obtain

$$\begin{aligned} |u_{x_0, \rho} - \ell(x_0)|^p &= \left| \int_{B_\rho(x_0)} (u - \ell(x_0) - D\ell(x - x_0)) dx \right|^p \\ &\leq \left(\int_{B_\rho(x_0)} |u - \ell| dx \right)^p \leq \int_{B_\rho(x_0)} |u - \ell|^p dx. \end{aligned}$$

The last term of (3.14) may estimate by using (3.12). This complete the proof. \square

Using Young's inequality and elementary calculations yield the following estimates.

Lemma 3.7. Consider fixed $a, b \geq 0$, $p \geq 1$. Then for any $\varepsilon > 0$, there exists $K = K(p, \varepsilon) \geq 0$ satisfying

$$(a + b)^p \leq (1 + \varepsilon)a^p + (1 + K)b^p. \quad (3.15)$$

Proof. We first consider the case $p = 2k - 1$ for $k \in \mathbb{N}$. By binomial theorem, we have

$$\begin{aligned} (a + b)^{2k-1} &= \sum_{m=0}^{2k-1} \binom{2k-1}{m} a^{2k-1-m} b^m \\ &= a^{2k-1} + b^{2k-1} + \sum_{m=0}^{k-1} \binom{2k-1}{m} (a^{2k-1-m} b^m + a^m b^{2k-1-m}). \end{aligned}$$

Using Young's inequality, we obtain

$$\sum_{m=0}^{k-1} \binom{2k-1}{m} (a^{2k-1-m} b^m + a^m b^{2k-1-m}) \leq \sum_{m=0}^{k-1} \binom{2k-1}{m} (\varepsilon' a^{2k-1} + C(k, m, \varepsilon') b^{2k-1}),$$

where $\varepsilon' > 0$ be fixed later. Thus, we get

$$\begin{aligned} (a+b)^{2k-1} &\leq a^{2k-1} + b^{2k-1} + \sum_{m=0}^{k-1} \binom{2k-1}{m} (\varepsilon' a^{2k-1} + C(k, m, \varepsilon') b^{2k-1}) \\ &= \left\{ 1 + \varepsilon' \sum_{m=0}^{k-1} \binom{2k-1}{m} \right\} a^{2k-1} + \left\{ 1 + \sum_{m=0}^{k-1} \binom{2k-1}{m} C(k, m, \varepsilon') \right\} b^{2k-1}. \end{aligned}$$

For any $\varepsilon > 0$ we conclude (3.7) by taking ε' as $\varepsilon = \varepsilon' \sum_{m=0}^{k-1} \binom{2k-1}{m}$.

In case of $p = 2k$, we may estimate similarly as above, hence we get

$$\begin{aligned} (a+b)^{2k} &= \sum_{m=0}^{2k} \binom{2k}{m} a^{2k-m} b^m \\ &= a^{2k} + b^{2k} + \sum_{m=0}^{k-1} \binom{2k}{m} (a^{2k-m} b^m + a^m b^{2k-m}) + \binom{2k}{k} a^k b^k \\ &\leq a^{2k} + b^{2k} + \sum_{m=0}^{k-1} \binom{2k}{m} (\varepsilon' a^{2k} + C(k, m, \varepsilon') b^{2k}) + \binom{2k}{k} \left(\varepsilon' a^{2k} + \frac{1}{\varepsilon'} b^{2k} \right) \\ &= \left\{ 1 + \varepsilon' \sum_{m=0}^k \binom{2k}{m} \right\} a^{2k} + \left\{ 1 + \sum_{m=0}^{k-1} \binom{2k}{m} C(k, m, \varepsilon') + \frac{1}{\varepsilon'} \right\} b^{2k}. \end{aligned}$$

This conclude that we have (3.7) for $p \in \mathbb{N}$.

For general $p \geq 1$, let $[p]$ be the greatest integer not greater than p . We write

$$(a+b)^p = (a+b)^{[p]} (a+b)^{p-[p]}.$$

By $0 \leq p - [p] < 1$, we have

$$(a+b)^{p-[p]} \leq a^{p-[p]} + b^{p-[p]}.$$

For $\varepsilon' > 0$ to be fixed later, we get

$$(a+b)^{[p]} \leq (1 + \varepsilon') a^{[p]} + (1 + K(p, \varepsilon')) b^{[p]},$$

since $[p] \in \mathbb{N}$. Combining two estimates, we obtain

$$\begin{aligned} (a+b)^p &\leq \left\{ (1 + \varepsilon') a^{[p]} + (1 + K(p, \varepsilon')) b^{[p]} \right\} (a^{p-[p]} + b^{p-[p]}) \\ &= (1 + \varepsilon') a^p + (1 + K(p, \varepsilon')) b^p + (1 + \varepsilon') a^{[p]} b^{p-[p]} + (1 + K(p, \varepsilon')) a^{p-[p]} b^{[p]} \\ &\leq (1 + \varepsilon') a^p + (1 + K(p, \varepsilon')) b^p + (1 + \varepsilon' + K(p, \varepsilon')) (a^{[p]} b^{p-[p]} + a^{p-[p]} b^{[p]}). \end{aligned}$$

Again for $\varepsilon'' > 0$ to be fixed later, by using Young's inequality, we conclude

$$(a+b)^p \leq (1 + \varepsilon') a^p + (1 + K(p, \varepsilon')) b^p + (1 + \varepsilon' + K(p, \varepsilon')) (\varepsilon'' a^p + C(p, \varepsilon'') b^p).$$

Take $\varepsilon' = \varepsilon/2$ and $\varepsilon'' = \varepsilon'/(1 + \varepsilon' + K(p, \varepsilon'))$, and this complete the proof. \square

Lemma 3.8 ([8, Lemma 2.1]). *For $\delta \geq 0$, and for all $a, b \in \mathbb{R}^k$ we have*

$$4^{-(1+2\delta)} \leq \frac{\int_0^1 (1 + |sa + (1-s)b|^2)^{\delta/2} ds}{(1 + |a|^2 + |b-a|^2)^{\delta/2}} \leq 4^\delta. \quad (3.16)$$

Remark 3.9. *Note that the above estimate is not critical. The left inequality could not take equal when $\delta = 0$.*

4 Proof of the main theorem

To obtain the regularity result (Theorem 2.2), we first prove Caccioppoli-type inequality. In the followings, we define $q > 0$ as a dual exponent of $p \geq 2$, that is, $q = p/(p-1)$. Here we note that $q \leq 2$.

Lemma 4.1. *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$ be a bounded weak solution of the elliptic system (1.1) under the structure condition **(H1)**, **(H2)**, **(H3)**, **(H4)** and **(H5)** with satisfying $\|u\|_\infty \leq M$ and $\lambda > 2^{(9p-10)/2}a(M)M$. For any $x_0 \in \Omega$ and $\rho \leq 1$ with $B_\rho(x_0) \Subset \Omega$, and any affine functions $\ell : \mathbb{R}^n \rightarrow \mathbb{R}^N$ with $|\ell(x_0)| \leq M$, we have the estimate*

$$\begin{aligned} & \int_{B_{\frac{\rho}{2}}(x_0)} \left\{ \frac{|Du - D\ell|^2}{(1 + |D\ell|)^2} + \frac{|Du - D\ell|^p}{(1 + |D\ell|)^p} \right\} dx \\ & \leq C_1 \left[\int_{B_\rho(x_0)} \left\{ \frac{|u - \ell|^2}{\rho^2(1 + |D\ell|)^2} + \frac{|u - \ell|^p}{\rho^p(1 + |D\ell|)^p} \right\} dx \right. \\ & \quad \left. + \omega \left(\int_{B_\rho(x_0)} |u - \ell(x_0)|^2 dx \right) + V(\rho) + (a^q |D\ell|^q + b^q) \rho^q \right], \end{aligned} \quad (4.1)$$

with the constant $C_1 = C_1(\lambda, p, L, a, M) \geq 1$.

Proof. Assume $x_0 \in \Omega$ and $\rho \leq 1$ satisfy $B_\rho(x_0) \Subset \Omega$. We take a standard cut-off function $\eta \in C_0^\infty(B_\rho(x_0))$ satisfying $0 \leq \eta \leq 1$, $|D\eta| \leq 4/\rho$, $\eta \equiv 1$ on $B_{\rho/2}(x_0)$. Then $\varphi := \eta^p(u - \ell)$ is admissible as a test function in (2.1), and we obtain

$$\begin{aligned} & \int_{B_\rho(x_0)} \eta^p \langle A(x, u, Du), Du - D\ell \rangle dx \\ & = - \int_{B_\rho(x_0)} \langle A(x, u, Du), p\eta^{p-1} D\eta \otimes (u - \ell) \rangle dx + \int_{B_\rho(x_0)} \langle f, \varphi \rangle dx. \end{aligned} \quad (4.2)$$

Furthermore, we have

$$\begin{aligned} & - \int_{B_\rho(x_0)} \eta^p \langle A(x, u, D\ell), Du - D\ell \rangle dx \\ & = \int_{B_\rho(x_0)} \langle A(x, u, D\ell), p\eta^{p-1} D\eta \otimes (u - \ell) \rangle dx - \int_{B_\rho(x_0)} \langle A(x, u, D\ell), D\varphi \rangle dx, \end{aligned} \quad (4.3)$$

and

$$\int_{B_\rho(x_0)} \langle (A(\cdot, \ell(x_0), D\ell))_{x_0, \rho}, D\varphi \rangle dx = 0. \quad (4.4)$$

Adding (4.2), (4.3) and (4.4), we obtain

$$\begin{aligned}
& \int_{B_\rho(x_0)} \eta^p \langle A(x, u, Du) - A(x, u, D\ell), Du - D\ell \rangle dx \\
&= - \int_{B_\rho(x_0)} \langle A(x, u, Du) - A(x, u, D\ell), p\eta^{p-1} D\eta \otimes (u - \ell) \rangle dx \\
&\quad - \int_{B_\rho(x_0)} \langle A(x, u, D\ell) - A(x, \ell(x_0), D\ell), D\varphi \rangle dx \\
&\quad - \int_{B_\rho(x_0)} \langle A(x, \ell(x_0), D\ell) - (A(\cdot, \ell(x_0), D\ell))_{x_0, \rho}, D\varphi \rangle dx \\
&\quad + \int_{B_\rho(x_0)} \langle f, \varphi \rangle dx \\
&=: \text{I} + \text{II} + \text{III} + \text{IV}.
\end{aligned} \tag{4.5}$$

The terms I, II, III, IV are defined above. Using the ellipticity condition **(H2)** to the left-hand side of (4.5), we get

$$\begin{aligned}
& \langle A(x, u, Du) - A(x, u, D\ell), Du - D\ell \rangle \\
&= \int_0^1 \langle D_\xi A(x, u, sDu + (1-s)D\ell)(Du - D\ell), Du - D\ell \rangle ds \\
&\geq \lambda |Du - D\ell|^2 \int_0^1 (1 + |sDu + (1-s)D\ell|)^{p-2} ds.
\end{aligned}$$

Then by using (3.16) in Lemma 3.8, we obtain

$$\begin{aligned}
& \langle A(x, u, Du) - A(x, u, D\ell), Du - D\ell \rangle \\
&\geq \lambda |Du - D\ell|^2 \int_0^1 (1 + |sDu + (1-s)D\ell|^2)^{(p-2)/2} ds \\
&\geq 2^{(12-9p)/2} \lambda \{ (1 + |D\ell|)^{p-2} |Du - D\ell|^2 + |Du - D\ell|^p \}.
\end{aligned} \tag{4.6}$$

For $\varepsilon > 0$ to be fixed later, using **(H1)** and Young's inequality, we have

$$\begin{aligned}
|\text{I}| &= \left| \int_{B_\rho(x_0)} \langle A(x, u, Du) - A(x, u, D\ell), p\eta^{p-1} D\eta \otimes (u - \ell) \rangle dx \right| \\
&\leq \int_{B_\rho(x_0)} p\eta^{p-1} \left| \int_0^1 D_\xi A(x, u, D\ell + s(Du - D\ell))(Du - D\ell) ds \right| |D\eta| |u - \ell| dx \\
&\leq \int_{B_\rho(x_0)} 2^{p-2} L p \eta^{p-1} \{ (1 + |D\ell|)^{p-2} + |Du - D\ell|^{p-2} \} |Du - D\ell| |D\eta| |u - \ell| dx \\
&\leq \int_{B_\rho(x_0)} (1 + |D\ell|)^{p-2} (\eta^{p-1} |Du - D\ell|) \left(2^p L p \left| \frac{u - \ell}{\rho} \right| \right) dx \\
&\quad + \int_{B_\rho(x_0)} (\eta^{p-1} |Du - D\ell|^{p-1}) \left(2^p L p \left| \frac{u - \ell}{\rho} \right| \right) dx \\
&\leq \varepsilon \int_{B_\rho(x_0)} \eta^p \{ (1 + |D\ell|)^{p-2} |Du - D\ell|^2 + |Du - D\ell|^p \} dx \\
&\quad + (2^p L p)^p \left(\frac{1}{\varepsilon} + \varepsilon^{-q/p} \right) \int_{B_\rho(x_0)} \left\{ (1 + |D\ell|)^{p-2} \left| \frac{u - \ell}{\rho} \right|^2 + \left| \frac{u - \ell}{\rho} \right|^p \right\} dx.
\end{aligned} \tag{4.7}$$

In order to estimate II, we use **(H3)**, $D\varphi = \eta^p(Du - \nu) + p\eta^{p-1}D\eta \otimes (u - \ell)$, and again Young's inequality, we get

$$\begin{aligned}
|\text{II}| &= \left| \int_{B_\rho(x_0)} \langle A(x, u, D\ell) - A(x, \ell(x_0), D\ell), D\varphi \rangle dx \right| \\
&\leq \varepsilon \int_{B_\rho(x_0)} \eta^p |Du - D\ell|^p dx + \varepsilon^{-q/p} \int_{B_\rho(x_0)} L^q \omega^q (|u - \ell(x_0)|^2) (1 + |D\ell|)^p dx \\
&\quad + \varepsilon \int_{B_\rho(x_0)} \left| \frac{u - \ell}{\rho} \right|^p dx + \varepsilon^{-q/p} \int_{B_\rho(x_0)} (4Lp)^q \omega^q (|u - \ell(x_0)|^2) (1 + |D\ell|)^p dx \\
&\leq \varepsilon \int_{B_\rho(x_0)} \eta^p |Du - D\ell|^p dx + \varepsilon \int_{B_\rho(x_0)} \left| \frac{u - \ell}{\rho} \right|^p dx \\
&\quad + 2(4Lp)^2 \varepsilon^{-q/p} (1 + |D\ell|)^p \omega \left(\int_{B_\rho(x_0)} |u - \ell(x_0)|^2 dx \right), \tag{4.8}
\end{aligned}$$

where we use Jensen's inequality in the last inequality. We next estimate III by using the VMO-condition **(H4)** and Young's inequality, we have

$$\begin{aligned}
|\text{III}| &= \left| \int_{B_\rho(x_0)} \langle A(x, \ell(x_0), D\ell) - (A(\cdot, \ell(x_0), D\ell))_{x_0, \rho}, D\varphi \rangle dx \right| \\
&\leq \frac{\varepsilon}{2^{p-1}} \int_{B_\rho(x_0)} \left\{ \eta^p |Du - D\ell| + \frac{4p|u - \ell|}{\rho} \right\}^p dx + \left(\frac{2^{p-1}}{\varepsilon} \right)^{q/p} \int_{B_\rho(x_0)} V_{x_0}^q(x, \rho) (1 + |D\ell|)^p dx.
\end{aligned}$$

Then using the fact that $V_{x_0}^q = V_{x_0}^{q-1} \cdot V_{x_0} \leq (2L)^{q-1} V_{x_0} \leq 2LV_{x_0}$, we infer

$$|\text{III}| \leq \varepsilon \int_{B_\rho(x_0)} \eta^p |Du - D\ell|^p dx + (4p)^p \varepsilon \int_{B_\rho(x_0)} \left| \frac{u - \ell}{\rho} \right|^p dx + 4L\varepsilon^{-q/p} (1 + |D\ell|)^p V(\rho). \tag{4.9}$$

For $\varepsilon' > 0$ to be fixed later, using **(H5)**, Lemma 3.7 and Young's inequality, we have

$$\begin{aligned}
|\text{IV}| &= \left| \int_{B_\rho(x_0)} \langle f, \varphi \rangle dx \right| \\
&\leq \int_{B_\rho(x_0)} a(|Du - D\ell| + |D\ell|)^p \eta^p |u - \ell| dx + \int_{B_\rho(x_0)} (b\eta\rho) \left| \frac{u - \ell}{\rho} \right| dx \\
&\leq \int_{B_\rho(x_0)} a\eta^p \{ (1 + \varepsilon') |Du - D\ell|^p + (1 + K(p, \varepsilon')) |D\ell|^p \} |u - \ell| dx + \varepsilon b^q \rho^q + \varepsilon^{-p/q} \int_{B_\rho(x_0)} \left| \frac{u - \ell}{\rho} \right|^p dx \\
&\leq a(1 + \varepsilon')(2M + |D\ell|\rho) \int_{B_\rho(x_0)} \eta^p |Du - D\ell|^p dx + 2\varepsilon^{-p/q} \int_{B_\rho(x_0)} \left| \frac{u - \ell}{\rho} \right|^p dx \\
&\quad + \varepsilon(1 + |D\ell|)^p \rho^q \{ a^q(1 + K)^q |D\ell|^q + b^q \}. \tag{4.10}
\end{aligned}$$

Combining (4.5), (4.8), (4.9) and (4.10), and set $\lambda' = 2^{(12-9p)/2} \lambda C$ $\Lambda := \lambda' - 3\varepsilon - a(1 + \varepsilon')(2M + |D\ell|\rho)$,

this gives

$$\begin{aligned}
& \Lambda \int_{B_\rho(x_0)} \eta^p \left\{ \frac{|Du - D\ell|^2}{(1 + |D\ell|)^2} + \frac{|Du - D\ell|^p}{(1 + |D\ell|)^p} \right\} dx \\
& \leq \left\{ \frac{(2^p Lp)^p}{\varepsilon} + 2(2^p Lp)^p \varepsilon^{-q/p} + 2(4p)^p \varepsilon + 2\varepsilon^{-p/q} \right\} \left[\int_{B_\rho(x_0)} \left\{ \left| \frac{u - \ell}{\rho(1 + |D\ell|)} \right|^2 + \left| \frac{u - \ell}{\rho(1 + |D\ell|)} \right|^p \right\} dx \right. \\
& \quad \left. + \omega \left(\int_{B_\rho(x_0)} |u - \ell(x_0)|^2 dx \right) + V(\rho) \right] + \varepsilon \{a^q(1 + K)^q |D\ell|^q + b^q\} \rho^q. \tag{4.11}
\end{aligned}$$

We now define

$$\begin{aligned}
c_1 &= \frac{144(2^p Lp)^2}{(\lambda' - 2aM)^2} + \frac{288(2^p Lp)^p}{(\lambda' - 2aM)^q} + 2(4p)^p + \frac{24^p}{(\lambda' - 2aM)^p}, \\
c_2 &= \begin{cases} \max \left\{ 1, \frac{3}{2aM} \frac{\lambda + 2aM}{\lambda - 2aM} \right\}, & (a \neq 0), \\ 1, & (a = 0), \end{cases}
\end{aligned}$$

and set $C_1 = 2^{2p+4} \{c_1 + (1 + K(p, \varepsilon'))^q\} c_2^q$. Here, $K(p, \varepsilon') \geq 0$ is a constant which we take in Lemma 3.7 as $\varepsilon' = (\lambda' - 2aM)/4aM$ at $a \neq 0$ and $K = 0$ at $a = 0$. Note that $c_1, c_2, C_1 \geq 1$.

We complete the proof by considering the following two cases.

In case of $a = 0$, choosing $\varepsilon = \lambda'/6$ and dividing (4.11) through by $\lambda'/2$ we obtain the desired estimate (4.1).

In the case $a \neq 0$, we set $\varepsilon = (\lambda' - 2aM)/12$ and $\varepsilon' = (\lambda' - 2aM)/4aM$. These choices imply

$$\Lambda = 3\varepsilon - \frac{\lambda' + 2aM}{4aM} a |D\ell| \rho.$$

In the sub-case $|D\ell| = 0$, dividing (4.11) through by 3ε we have

$$\begin{aligned}
& \int_{B_\rho(x_0)} \eta^p \{ |Du|^2 + |Du|^p \} dx \\
& \leq \left\{ \frac{48(2^p Lp)^p}{(\lambda' - 2aM)^2} + \frac{96(2^p Lp)^p}{(\lambda' - 2aM)^q} + \frac{2(4p)^p}{3} + \frac{96}{(\lambda' - 2aM)^p} \right\} \\
& \quad \times \left[\int_{B_\rho(x_0)} \left\{ \left| \frac{u - \ell(x_0)}{\rho} \right|^2 + \left| \frac{u - \ell(x_0)}{\rho} \right|^p \right\} dx + \omega \left(\int_{B_\rho(x_0)} |u - \ell(x_0)|^2 dx \right) + V(\rho) + \varepsilon + b^q \rho^q \right].
\end{aligned}$$

In remaining sub-case $|D\ell| \neq 0$, we first assume $\rho \leq \max \left\{ 1, \frac{1}{c_2 a |D\ell|} \right\}$. Then $\Lambda \geq \varepsilon$ and dividing (4.11) through by ε we immediately obtain (4.1).

Finally, for $D\ell \neq 0$, $\frac{1}{c_2 a |D\ell|} \leq \rho \leq 1$ we use the result of the sub-case $D\ell = 0$ and the definition of ρ_0 ,

to obtain

$$\begin{aligned}
& \int_{B_{\frac{\rho}{2}}(x_0)} \left\{ (1 + |D\ell|)^{p-2} |Du - D\ell|^2 + |Du - D\ell|^p \right\} dx \\
& \leq 2^{p+1} \int_{B_{\frac{\rho}{2}}(x_0)} (|Du|^2 + |Du|^p) dx + 2^{p+1} (|D\ell|^2 + |D\ell|^p) \\
& \leq 2^{p+1} c_1 \left[\int_{B_{\rho}(x_0)} \left\{ \left| \frac{u - \ell(x_0)}{\rho} \right|^2 + \left| \frac{u - \ell(x_0)}{\rho} \right|^p \right\} dx + \omega \left(\int_{B_{\rho}(x_0)} |u - \ell(x_0)|^2 dx \right) + V(\rho) + b^q \rho^q \right] \\
& \quad + 2^{p+1} \{ (1 + |D\ell|)^2 + (1 + |D\ell|)^p \} \\
& \leq 2^{2p+4} c_1 \left[\int_{B_{\rho}(x_0)} \left\{ \left| \frac{u - \ell}{\rho} \right|^2 + \left| \frac{u - \ell}{\rho} \right|^p \right\} dx + \omega \left(\int_{B_{\rho}(x_0)} |u - \ell(x_0)|^2 dx \right) + V(\rho) + (1 + |D\ell|)^p + b^q \rho^q \right] \\
& \leq 2^{2p+4} c_1 c_2^q \left[\int_{B_{\rho}(x_0)} \left\{ (1 + |D\ell|)^{p-2} \left| \frac{u - \ell}{\rho} \right|^2 + \left| \frac{u - \ell}{\rho} \right|^p \right\} dx \right. \\
& \quad \left. + (1 + |D\ell|)^p \omega \left(\int_{B_{\rho}(x_0)} |u - \ell(x_0)|^2 dx \right) + (1 + |D\ell|)^p V(\rho) + (1 + |D\ell|)^p (a^q |D\ell|^q + b^q) \rho^q \right]
\end{aligned}$$

and the assertion also follows in this case. \square

Remark 4.2. As we shown in (4.6), the hypothesis $\lambda > 2^{(9p-10)/2} a(M)M$ is necessary to estimate the left-hand side of (4.5) from below. On the other hand, when $p = 2$, we don't have to use Lemma 3.8 in estimating the left-hand side of (4.5) because the term $(1 + |sDu + (1-s)v|)^{p-2}$ vanish. Thus we only need $\lambda > 2a(M)M$ to get Caccioppoli-type inequality. The gap between $p > 2$ and $p = 2$ could not avoid. As we need the estimate of type $|a + b|^{p-2} \geq C(p)(|a|^{p-2} + |b|^{p-2})$ to prove Caccioppoli-type inequality, the constant must be $C(2) \leq 1/2$ and we could not take $C(2) = 1$.

To use the \mathcal{A} -harmonic approximation lemma, we need to estimate $\int_{B_{\rho}(x_0)} \mathcal{A}(D(u - \ell), D\varphi) dx$.

Lemma 4.3. Assume the same assumption in Lemma 4.1. Then for any $x_0 \in \Omega$ and $\rho \leq \rho_0$ satisfy $B_{2\rho}(x_0) \Subset \Omega$, and any affine functions $\ell : \mathbb{R}^n \rightarrow \mathbb{R}^N$ with $|\ell(x_0)| \leq M$, the inequality

$$\begin{aligned}
\int_{B_{\rho}(x_0)} \mathcal{A}(Dv, D\varphi) dx & \leq C_2 (1 + |D\ell|) \left[\mu^{1/2} \left(\sqrt{\Psi_*(x_0, 2\rho, \ell)} \right) \sqrt{\Psi_*(x_0, 2\rho, \ell)} \right. \\
& \quad \left. + \Psi_*(x_0, 2\rho, \ell) + \rho(a|D\ell|^p + b) \right] \sup_{B_{\rho}(x_0)} |D\varphi| \quad (4.12)
\end{aligned}$$

holds for all $\varphi \in C_0^\infty(B_{\rho}(x_0), \mathbb{R}^N)$ and a constant $C_2 = C_2(n, \lambda, L, p, a(M)) \geq 1$, where

$$\begin{aligned}
\mathcal{A}(Dv, D\varphi) & := \frac{1}{(1 + |D\ell|)^{p-1}} \left\langle (D_\xi A(\cdot, \ell(x_0), D\ell))_{x_0, \rho} Dv, D\varphi \right\rangle, \\
\Phi(x_0, \rho, \ell) & := \int_{B_{\rho}(x_0)} \left\{ \frac{|Du - D\ell|^2}{(1 + |D\ell|)^2} + \frac{|Du - D\ell|^p}{(1 + |D\ell|)^p} \right\} dx, \\
\Psi(x_0, \rho, \ell) & := \int_{B_{\rho}(x_0)} \left\{ \frac{|u - \ell|^2}{\rho^2 (1 + |D\ell|)^2} + \frac{|u - \ell|^p}{\rho^p (1 + |D\ell|)^p} \right\} dx, \\
\Psi_*(x_0, \rho, \ell) & := \Psi(x_0, \rho, \ell) + \omega \left(\int_{B_{\rho}(x_0)} |u - \ell(x_0)|^2 dx \right) + V(\rho) + (a^q |D\ell|^q + b^q) \rho^q, \\
v & := u - \ell = u - \ell(x_0) - D\ell(x - x_0).
\end{aligned}$$

Proof. Assume $x_0 \in \Omega$ and $\rho \leq 1$ satisfy $B_{2\rho}(x_0) \Subset \Omega$. Without loss of generality we may assume $\sup_{B_\rho(x_0)} |D\varphi| \leq 1$. Note $\sup_{B_\rho(x_0)} |\varphi| \leq \rho \leq 1$. Using the fact that $\int_{B_\rho(x_0)} A(x_0, \xi, \nu) D\varphi dx = 0$, we deduce

$$\begin{aligned}
& (1 + |D\ell|)^{p-1} \int_{B_\rho(x_0)} \mathcal{A}(Dv, D\varphi) dx \\
&= \int_{B_\rho(x_0)} \int_0^1 \left\langle \left[(D_\xi A(\cdot, \ell(x_0), D\ell))_{x_0, \rho} - (D_\xi A(\cdot, \ell(x_0), D\ell + sDv))_{x_0, \rho} \right] Dv, D\varphi \right\rangle ds dx \\
&\quad + \int_{B_\rho(x_0)} \left\langle (A(\cdot, \ell(x_0), Du))_{x_0, \rho} - A(x, \ell(x_0), Du), D\varphi \right\rangle dx \\
&\quad + \int_{B_\rho(x_0)} \langle A(x, \ell(x_0), Du) - A(x, u, Du), D\varphi \rangle dx \\
&\quad + \int_{B_\rho(x_0)} \langle f, \varphi \rangle dx \\
&=: \text{I} + \text{II} + \text{III} + \text{IV}
\end{aligned} \tag{4.13}$$

where terms I, II, III, IV are define above.

Using the modulus of continuity μ from **(H1)**, Jensen's inequality and Hölder's inequality, we estimate

$$\begin{aligned}
|\text{I}| &= \left| \int_{B_\rho(x_0)} \int_0^1 \left\langle \left[(D_\xi A(\cdot, \ell(x_0), D\ell))_{x_0, \rho} - (D_\xi A(\cdot, \ell(x_0), D\ell + sDv))_{x_0, \rho} \right] Dv, D\varphi \right\rangle ds dx \right| \\
&\leq 2^{p-2} L \int_{B_\rho(x_0)} \int_0^1 \mu \left(\frac{|Du - D\ell|}{1 + |D\ell|} \right) (1 + |D\ell| + |Du - D\ell|)^{p-2} |Du - D\ell| |D\varphi| ds dx \\
&\leq 2^{2p-4} L (1 + |D\ell|)^{p-1} \int_{B_\rho(x_0)} \mu \left(\frac{|Du - D\ell|}{1 + |D\ell|} \right) \left\{ \frac{|Du - D\ell|}{1 + |D\ell|} + \frac{|Du - D\ell|^{p-1}}{(1 + |D\ell|)^{p-1}} \right\} dx \\
&\leq 2^{2p-4} L (1 + |D\ell|)^{p-1} \left[\mu^{1/2} \left(\sqrt{\Phi(x_0, \rho, \ell)} \right) \sqrt{\Phi(x_0, \rho, \ell)} + \mu^{1/p} \left(\Phi^{1/2}(x_0, \rho, \ell) \right) \Phi^{1/q}(x_0, \rho, \ell) \right] \\
&\leq 2^{2p-3} L (1 + |D\ell|)^{p-1} \left[\mu^{1/2} \left(\sqrt{\Phi(x_0, \rho, \ell)} \right) \sqrt{\Phi(x_0, \rho, \ell)} + \Phi(x_0, \rho, \ell) \right].
\end{aligned} \tag{4.14}$$

The last inequality follows from the fact that $a^{1/p} b^{1/q} = a^{1/p} b^{1/p} b^{(p-2)/p} \leq a^{1/2} b^{1/2} + b$ holds by Young's inequality.

By using the VMO-condition, Young's inequality and the bound $V_{x_0}(x, \rho) \leq 2L$, the term II can be estimated as

$$\begin{aligned}
|\text{II}| &= \left| \int_{B_\rho(x_0)} \left\langle (A(\cdot, \ell(x_0), Du))_{x_0, \rho} - A(x, \ell(x_0), Du), D\varphi \right\rangle dx \right| \\
&\leq 2^{p-2} (1 + |D\ell|)^{p-1} \int_{B_\rho(x_0)} \left\{ V_{x_0}(x, \rho) + V_{x_0}(x, \rho) \frac{|Du - D\ell|^{p-1}}{(1 + |D\ell|)^{p-1}} \right\} dx \\
&\leq 2^{p-2} (1 + |D\ell|)^{p-1} \left[(1 + (2L)^{p-1}) V(\rho) + \Phi(x_0, \rho, \ell) \right].
\end{aligned} \tag{4.15}$$

Similarly, we estimate the term III by using the continuity condition **(H3)**, Young's inequality, the

bound $\omega \leq 1$ and Jensen's inequality. This leads us to

$$\begin{aligned}
|\text{III}| &= \left| \int_{B_\rho(x_0)} \langle A(x, \ell(x_0), Du) - A(x, u, Du), D\varphi \rangle dx \right| \\
&\leq L \int_{B_\rho(x_0)} (1 + |D\ell| + |Du - D\ell|)^{p-1} \omega(|u - \ell(x_0)|^2) dx \\
&\leq 2^{p-1} L (1 + |D\ell|)^{p-1} \left[\omega \left(\int_{B_\rho(x_0)} |u - \ell(x_0)|^2 dx \right) + \Phi(x_0, \rho, \ell) \right]. \tag{4.16}
\end{aligned}$$

By using the growth condition **(H5)** and $\sup_{B_\rho(x_0)} |\varphi| \leq \rho \leq 1$, we have

$$\begin{aligned}
|\text{IV}| &= \left| \int_{B_\rho(x_0)} \langle f, \varphi \rangle dx \right| \\
&\leq \int_{B_\rho(x_0)} \rho(a|Du|^p + b) dx \\
&\leq 2^{p-1} a (1 + |D\ell|)^p \Phi(x_0, \rho, \ell) + 2^{p-1} \rho (1 + |D\ell|)^{p-1} (a|D\ell|^p + b). \tag{4.17}
\end{aligned}$$

Combining (4.13) with the estimates (4.14), (4.15), (4.16) and (4.17), and set $C_2 = 2^{n+2p}(1 + a + (2L)^{p-1})C_1 \geq 1$, we finally arrive at

$$\begin{aligned}
&\int_{B_\rho(x_0)} \mathcal{A}(Dv, D\varphi) dx \\
&\leq 2^{2p-1} (1 + a + (2L)^{p-1}) (1 + |D\ell|) \\
&\quad \times \left[\mu^{1/2} \left(\sqrt{\Phi(x_0, \rho, \ell)} \right) \sqrt{\Phi(x_0, \rho, \ell)} + \Phi(x_0, \rho, \ell) + \Psi_*(x_0, \rho, \ell) + \rho(a|D\ell|^p + b) \right] \\
&\leq C_2 (1 + |D\ell|) \left[\mu^{1/2} \left(\sqrt{\Psi_*(x_0, 2\rho, \ell)} \right) \sqrt{\Psi_*(x_0, 2\rho, \ell)} + \Psi_*(x_0, 2\rho, \ell) + \rho(a|D\ell|^p + b) \right],
\end{aligned}$$

where we use Caccioppoli-type inequality (Lemma 4.1), $\Phi(x_0, \rho, \ell) \leq C_1 \Psi_*(x_0, 2\rho, \ell)$ and the concavity of μ to have $\mu(cs) \leq c\mu(s)$ for $c \geq 1$ at the last step. \square

From now on, we write $\Phi(\rho) = \Phi(x_0, \rho, \ell_{x_0, \rho})$, $\Psi(\rho) = \Psi(x_0, \rho, \ell_{x_0, \rho})$, $\Psi_*(\rho) = \Psi_*(x_0, \rho, \ell_{x_0, \rho})$ for $x_0 \in \Omega$ and $0 < \rho \leq 1$. Here $\ell_{x_0, \rho}$ is a minimizer of (3.9).

Now we are in the position to establish the excess improvement.

Lemma 4.4. *Assume the same assumptions with Lemma 4.3. Let $\theta \in (0, 1/4]$ be arbitrary and impose the following smallness conditions on the excess:*

$$(i) \quad \mu^{1/2} \left(\sqrt{\Psi_*(\rho)} \right) + \sqrt{\Psi_*(\rho)} \leq \frac{\delta}{2} \text{ with the constant } \delta = \delta(n, N, p, \lambda, L, \theta^{n+p+2}) \text{ from Lemma 3.2,}$$

$$(ii) \quad \Psi(\rho) \leq \frac{\theta^{n+2}}{4n(n+2)},$$

$$(iii) \quad \gamma(\rho) := \left[\Psi_*^{q/2}(\rho) + \delta^{-q} \rho^q (a|D\ell_{x_0, \rho}| + b)^q \right]^{1/q} \leq 1.$$

Then there holds the excess improvement estimate

$$\Psi(\theta\rho) \leq C_3 \theta^2 \Psi_*(\rho) \tag{4.18}$$

with a constant $C_3 \geq 1$ that depends only on $n, N, \lambda, L, p, a, M$ and θ .

Proof. We first rescale u and set

$$w := \frac{u - \ell_{x_0, \rho}}{C_2(1 + |D\ell_{x_0, \rho}|)\gamma}.$$

We claim that w satisfies the assumptions of Lemma 3.2. By Lemma 4.3, with $\rho/2$ and $\ell_{x_0, \rho}$ instead of ρ and ℓ , and assumption (i), the map w is approximately \mathcal{A} -harmonic in the sense that

$$\begin{aligned} \int_{B_{\rho/2}(x_0)} \mathcal{A}(Dw, D\varphi) dx &\leq \left[\mu^{1/2} \left(\sqrt{\Psi_*(\rho)} \right) + \sqrt{\Psi_*(\rho)} + \frac{\delta}{2} \right] \sup_{B_{\rho/2}(x_0)} |D\varphi| \\ &\leq \delta \sup_{B_{\rho/2}(x_0)} |D\varphi|, \end{aligned}$$

for all $\varphi \in C_0^\infty(B_{\rho/2}(x_0), \mathbb{R}^N)$, with the constant δ determined by Lemma 3.2 for the choice $\varepsilon = \theta^{n+p+2}$. Moreover, the choice of C_2 and the Caccioppoli-type inequality (Lemma 4.1) infer

$$\int_{B_{\rho/2}(x_0)} \{|Dw|^2 + \gamma^{p-2}|Dw|^p\} dx \leq \frac{C_1 \Psi_*(\rho)}{C_2^2 \gamma^2} \leq \frac{C_1}{C_2^2} \leq 1.$$

Thus, Lemma 3.2 ensures the existence of an \mathcal{A} -harmonic map h with the properties

$$\int_{B_{\rho/2}(x_0)} \left\{ \left| \frac{w-h}{\rho/2} \right|^2 + \gamma^{p-2} \left| \frac{w-h}{\rho/2} \right|^p \right\} dx \leq \theta^{n+p+2}, \quad (4.19)$$

$$\int_{B_{\rho/2}(x_0)} \{|Dh|^2 + \gamma^{p-2}|Dh|^p\} dx \leq c(n, p). \quad (4.20)$$

Since h is \mathcal{A} -harmonic, Theorem 3.3 yields the estimate

$$\sup_{B_{\rho/4}(x_0)} |D^2 h|^2 \leq C_0 c(n, p) \left(\frac{\rho}{2} \right)^{-2}.$$

From this we infer the estimate for $s = 2$ as well as for $s = p$

$$\sup_{B_{\rho/4}(x_0)} |D^2 h|^s \leq (C_0 c(n, p))^{s/2} \left(\frac{\rho}{2} \right)^{-s}.$$

Therefore, using Taylor's theorem, we have the decay estimate, where $\theta \in (0, 1/4]$ can be chosen arbitrarily:

$$\begin{aligned} &\gamma^{s-2}(\theta\rho)^{-s} \int_{B_{\theta\rho}(x_0)} |w - h(x_0) - Dh(x_0)(x - x_0)|^s dx \\ &\leq 2^{s-1} \gamma^{s-2}(\theta\rho)^{-s} \left[\int_{B_{\theta\rho}(x_0)} |w - h|^s dx + \int_{B_{\theta\rho}(x_0)} |h(x) - h(x_0) - Dh(x_0)(x - x_0)|^s dx \right] \\ &\leq \left\{ 2^{-n-1} + 2^{2s-1} (C_0 c(n, p))^{s/2} \right\} \theta^2. \end{aligned}$$

Here we applied the energy bound (4.19) for the last estimate. Set $C(s) = \tilde{c}(n, s) C_2^s (2^{-n-1} +$

$2^{2s-1}(C_0 c(n, p))^{s/2}(1 + 2^{2/p}\delta^{-q})$ where $\tilde{c}(n, s)$ is a constant from Lemma 3.6 and we conclude

$$\begin{aligned}
& (\theta\rho)^{-s} \int_{B_{\theta\rho}(x_0)} |u - \ell_{x_0, \theta\rho}|^s dx \\
& \leq \tilde{c}(n, s) (\theta\rho)^{-s} \int_{B_{\theta\rho}(x_0)} |u - \ell_{x_0, \rho} - C_2\gamma(1 + |D\ell_{x_0, \rho}|)(h(x_0) + Dh(x_0)(x - x_0))|^s dx \\
& = \tilde{c}(n, s) C_2^s (\theta\rho)^{-s} \gamma^s (1 + |D\ell_{x_0, \rho}|)^s \int_{B_{\theta\rho}(x_0)} |w - h(x_0) - Dh(x_0)(x - x_0)|^s dx \\
& \leq \tilde{c}(n, s) C_2^s (2^{-n-1} + 2^{2s-1}(C_0 c(n, p))^{s/2}) \gamma^2 (1 + |D\ell_{x_0, \rho}|)^s \theta^2 \\
& \leq \tilde{c}(n, s) C_2^s (2^{-n-1} + 2^{2s-1}(C_0 c(n, p))^{s/2}) (1 + |D\ell_{x_0, \rho}|)^s \theta^2 \left[\Psi_*^{q/2}(\rho) + 2^{q/p}\delta^{-q}\Psi_*(\rho) \right]^{2/q} \\
& \leq C(s)(1 + |D\ell_{x_0, \rho}|)^s \theta^2 \Psi_*(\rho). \tag{4.21}
\end{aligned}$$

Here we would like to replace the term $|D\ell_{x_0, \rho}|$ on the right-hand side by $|D\ell_{x_0, \theta\rho}|$. For this, we use (3.11) and the assumption (ii) in order to estimate

$$\begin{aligned}
|D\ell_{x_0, \rho} - D\ell_{x_0, \theta\rho}|^2 & \leq \frac{n(n+2)}{(\theta\rho)^2} \int_{B_{\theta\rho}(x_0)} |u - \ell_{x_0, \rho}|^2 dx \\
& \leq \frac{n(n+2)}{\theta^{n+2}\rho^2} \int_{B_\rho(x_0)} |u - \ell_{x_0, \rho}|^2 dx \\
& \leq \frac{n(n+2)}{\theta^{n+2}} (1 + |D\ell_{x_0, \rho}|)^2 \Psi(\rho) \leq \frac{1}{4} (1 + |D\ell_{x_0, \rho}|)^2.
\end{aligned}$$

This yields

$$1 + |D\ell_{x_0, \rho}| \leq 1 + |D\ell_{x_0, \theta\rho}| + |D\ell_{x_0, \rho} - D\ell_{x_0, \theta\rho}| \leq 1 + |D\ell_{x_0, \theta\rho}| + \frac{1}{2}(1 + |D\ell_{x_0, \rho}|),$$

and after reabsorbing the last term from the right-hand side on the left, we also obtain

$$1 + |D\ell_{x_0, \rho}| \leq 2(1 + |D\ell_{x_0, \theta\rho}|).$$

Plugging this into (4.21), we deduce

$$(\theta\rho)^{-s} \int_{B_{\theta\rho}(x_0)} |u - \ell_{x_0, \theta\rho}|^s dx \leq C(s)(1 + |D\ell_{x_0, \theta\rho}|)^s \theta^2 \Psi_*(\rho) \quad \text{for } s = 2 \text{ and } s = p.$$

Finally, set $C_3 = C(2) + C(p) \geq 1$ and dividing by $(1 + |D\ell_{x_0, \theta\rho}|)^s$, then adding the corresponding terms for $s = 2$ and $s = p$, we deduce the claim. \square

We fix an arbitrarily Hölder exponent $\alpha \in (0, 1)$ and define the Campanato-type excess

$$C_\alpha(x_0, \rho) := \rho^{-2\alpha} \int_{B_\rho(x_0)} |u - u_{x_0, \rho}|^2 dx.$$

In the following lemma, we iterate the excess improvement estimate from Lemma 4.4.

Lemma 4.5. *Under the same assumption with Lemma 4.4, for every $\alpha \in (0, 1)$, there exists constants $\varepsilon_*, \kappa_*, \rho_* > 0$ and $\theta \in (0, 1/8]$, all depending at most on $n, N, \lambda, p, L, \alpha, \rho_0, \mu(\cdot), \omega(\cdot), V(\cdot), a, b$ and M , such that the conditions*

$$\Psi(\rho) < \varepsilon_*, \quad \text{and} \quad C_\alpha(x_0, \rho) < \kappa_* \tag{A_0}$$

for all $\rho \in (0, \rho_*)$ with $B_\rho(x_0) \Subset \Omega$, imply

$$\Psi(\theta^k \rho) < \varepsilon_*, \quad \text{and} \quad C_\alpha(x_0, \theta^k \rho) < \kappa_* \quad (A_k)$$

respectively, for every $k \in \mathbb{N}$.

Proof. We begin by choosing the constants. First, let

$$\theta := \min \left\{ \left(\frac{1}{16n(n+2)} \right)^{1/(2-2\alpha)}, \frac{1}{\sqrt{4C_3}} \right\} \leq \frac{1}{8},$$

with the constant C_3 determined in Lemma 4.3. In particular, the choice of $\theta = \theta(n, N, \lambda, L, a, M, \alpha) > 0$ fixes the constant $\delta = \delta(n, N, \lambda, L, a, M, \alpha) > 0$ from Lemma 3.2. Next, we fix an $\varepsilon_* = \varepsilon_*(n, N, \lambda, L, a, M, \alpha, \mu(\cdot)) > 0$ sufficiently small to ensure

$$\varepsilon_* \leq \frac{\theta^{n+2}}{16n(n+2)} \quad \text{and} \quad \mu^{1/2}(\sqrt{4\varepsilon_*}) + \sqrt{4\varepsilon_*} \leq \frac{\delta}{2}.$$

Then, we choose $\kappa_* = \kappa_*(n, N, \lambda, L, a, M, \alpha, \mu(\cdot), \omega(\cdot)) > 0$ so small that

$$\omega(\kappa_*) < \varepsilon_*.$$

Finally, we fix $\rho_* = \rho_*(n, N, \lambda, p, L, \alpha, \rho_0, \mu(\cdot), \omega(\cdot), V(\cdot), a, b, M) > 0$ small enough to guarantee

$$\rho_* \leq \min\{\rho_0, \kappa_*^{1/(2-2\alpha)}, 1\}, \quad V(\rho_*) < \varepsilon_* \quad \text{and} \quad \left\{ \left(a\sqrt{n(n+2)\kappa_*} \right)^q + b^q \right\} \rho_*^{q\alpha} < \varepsilon_*.$$

Now we prove the assertion (A_k) by induction. We assume that we have already established (A_k) up to some $k \in \mathbb{N} \cup \{0\}$. We begin with proving the first part of the assertion (A_{k+1}) , that is, the one concerning $\Psi(\theta^{k+1}\rho)$. First, using (3.12) with $\ell \equiv u_{x_0, \theta^k \rho}$, we obtain

$$\begin{aligned} |D\ell_{x_0, \theta^k \rho}|^2 &\leq \frac{n(n+2)}{(\theta^k \rho)^2} \int_{B_{\theta^k \rho}(x_0)} |u - u_{x_0, \theta^k \rho}|^2 dx \\ &= n(n+2)(\theta^k \rho)^{2\alpha-2} C_\alpha(x_0, \theta^k \rho) \\ &\leq n(n+2)\rho_*^{2\alpha-2} \kappa_*. \end{aligned} \quad (4.22)$$

Thus, the assumption (A_k) , the choice of κ_* and ρ_* , and the above estimate infer

$$\begin{aligned} \Psi_*(\theta^k \rho) &\leq \Psi(\theta^k \rho) + \omega(C_\alpha(x_0, \theta^k \rho)) + V(\theta^k \rho) + (a^q |D\ell_{x_0, \theta^k \rho}|^q + b^q)(\theta^k \rho)^q \\ &\leq \varepsilon_* + \omega(\kappa_*) + V(\rho_*) + \left(\left(a\sqrt{n(n+2)\kappa_*} \right)^q + b^q \right) \rho_*^{q\alpha} < 4\varepsilon_*. \end{aligned} \quad (4.23)$$

Now it is easy to check that our choice of ε_* implies that the smallness condition assumptions (i) and (ii) in Lemma 4.4 are satisfied on the level $\theta^k \rho$, that is, we have

$$\mu^{1/2} \left(\sqrt{\Psi_*(\theta^k \rho)} \right) + \sqrt{\Psi_*(\theta^k \rho)} < \mu^{1/2}(\sqrt{4\varepsilon_*}) + \sqrt{4\varepsilon_*} \leq \frac{\delta}{2}, \quad (4.24)$$

and

$$\Psi(\theta^k \rho) < \varepsilon_* < \frac{\theta^{n+2}}{4n(n+2)}. \quad (4.25)$$

Furthermore, we have the smallness condition assumption (iii), that is,

$$\gamma(\theta^k \rho) = \left[\Psi_*^{q/2}(\theta^k \rho) + \delta^{-q}(\theta^k \rho)^q (a |D\ell_{x_0, \theta^k \rho}| + b)^q \right]^{1/q} \leq 1. \quad (4.26)$$

To check (4.26), first, note that $\Psi_*(\theta^k \rho) < 1$ holds by the estimate (4.23) and the choice of ε_* . This implies

$$\Psi_*^{q/2}(\theta^k \rho) \leq \Psi_*^{1/2}(\theta^k \rho) < \sqrt{4\varepsilon_*} \leq \frac{\delta}{4}. \quad (4.27)$$

Next, using (4.22) and $\rho_*^{\alpha-1} \geq 1$, we obtain

$$\begin{aligned} \delta^{-q}(\theta^k \rho)^q (a|D\ell|_{x_0, \theta^k \rho} + b)^q &\leq \delta^{-q} \rho_*^q (a\sqrt{n(n+2)\kappa_*} \rho_*^{\alpha-1} + b)^q \\ &\leq \delta^{-q} \rho_*^{q\alpha} (a\sqrt{n(n+2)\kappa_*} + b)^q \\ &\leq \delta^{-q} \rho_*^{q\alpha} 2^{q/p} \left\{ \left(a\sqrt{n(n+2)\kappa_*} \right)^q + b^q \right\}. \end{aligned}$$

Then the choice of ρ_* and ε_* imply

$$\delta^{-q}(\theta^k \rho)^q (a|D\ell|_{x_0, \theta^k \rho} + b)^q \leq \delta^{-q} 2^{q/p} \varepsilon_* \leq 2^{-4+q/p} \delta^{2-q} \leq \frac{\delta}{8}. \quad (4.28)$$

Therefore combining (4.27) and (4.28), we have (4.26). We may thus apply Lemma 4.4 with the radius $\theta^k \rho$ instead of ρ , which yields

$$\Psi(\theta^{k+1} \rho) \leq C_3 \theta^2 \Psi_*(\theta^k \rho) < 4C_3 \theta^2 \varepsilon_* \leq \varepsilon_*,$$

by the choice of θ . We have thus established the first part of the assertion (A_{k+1}) and it remains to prove the second one, that is, the one concerning $C_\alpha(x_0, \theta^{k+1} \rho)$. For this aim, we first compute

$$\frac{1}{(\theta^k \rho)^2} \int_{B_{\theta^k \rho}(x_0)} |u - \ell_{x_0, \theta^k \rho}|^2 dx \leq (1 + |D\ell_{x_0, \theta^k \rho}|)^2 \Psi(\theta^k \rho) \leq 2\varepsilon_* + 2\varepsilon_* |D\ell_{x_0, \theta^k \rho}|^2$$

where we used the assumption (A_k) in the last step. Since $\ell_{x_0, \theta^k \rho}(x) = u_{x_0, \theta^k \rho} + D\ell_{x_0, \theta^k \rho}(x - x_0)$, we can estimate

$$\begin{aligned} C_\alpha(x_0, \theta^{k+1} \rho) &\leq (\theta^{k+1} \rho)^{-2\alpha} \int_{B_{\theta^{k+1} \rho}(x_0)} |u - u_{x_0, \theta^k \rho}|^2 dx \\ &\leq 2(\theta^{k+1} \rho)^{-2\alpha} \left[\int_{B_{\theta^{k+1} \rho}(x_0)} |u - \ell_{x_0, \theta^k \rho}|^2 dx + |D\ell_{x_0, \theta^k \rho}|^2 (\theta^{k+1} \rho)^2 \right] \\ &\leq 2(\theta^{k+1} \rho)^{-2\alpha} \left[\theta^{-n} \int_{B_{\theta^k \rho}(x_0)} |u - \ell_{x_0, \theta^k \rho}|^2 dx + |D\ell_{x_0, \theta^k \rho}|^2 (\theta^{k+1} \rho)^2 \right] \\ &\leq 4(\theta^k \rho)^{2-2\alpha} [\varepsilon_* \theta^{-n-2\alpha} + |D\ell_{x_0, \theta^k \rho}|^2 (\varepsilon_* \theta^{-n-2\alpha} + \theta^{2-2\alpha})]. \end{aligned}$$

Using (4.22) and recalling the choice of ρ_* , ε_* and θ , we deduce

$$\begin{aligned} C_\alpha(x_0, \theta^{k+1} \rho) &\leq 4\rho_*^{2-2\alpha} [\varepsilon_* \theta^{-n-2\alpha} + n(n+2)\kappa_* \rho_*^{2-2\alpha} (\varepsilon_* \theta^{-n-2\alpha} + \theta^{2-2\alpha})] \\ &\leq \frac{1}{4} \rho_*^{2-2\alpha} \theta^{2-2\alpha} + 8n(n+2)\kappa_* \theta^{2-2\alpha} \\ &\leq \frac{1}{4} \kappa_* + \frac{1}{2} \kappa_* < \kappa_*. \end{aligned}$$

This proves the second part of the assertion (A_{k+1}) and finally we conclude the proof of the lemma. \square

Now we are in position to prove Theorem 2.2.

Proof of Theorem 2.2. By Lebesgue's differentiation theorem, it holds $\mathcal{L}^n(\Omega \setminus \Omega_u) = 0$. Consequently, it suffices to show that every $x_0 \in \Omega \setminus (\Sigma_1 \cup \Sigma_2)$ is a regular point. We first note that for every $0 < \rho < \text{dist}(x_0, \partial\Omega)$, the bound (3.12) and the Poincaré inequality (Lemma 3.4) imply

$$\begin{aligned} |D\ell_{x_0, \rho} - (Du)_{x_0, \rho}|^2 &\leq \frac{n(n+2)}{\rho^2} \int_{B_\rho(x_0)} |u - u_{x_0, \rho} - (Du)_{x_0, \rho}(x - x_0)|^2 dx \\ &\leq C_P n(n+2) \int_{B_\rho(x_0)} |Du - (Du)_{x_0, \rho}|^2 dx. \end{aligned}$$

Consequently, by another application of the Poincaré inequality and (3.12), we obtain

$$\begin{aligned} \Psi(x_0, \rho, \ell_{x_0, \rho}) &\leq \int_{B_\rho(x_0)} \left\{ \left| \frac{u - \ell_{x_0, \rho}}{\rho} \right|^2 + \left| \frac{u - \ell_{x_0, \rho}}{\rho} \right|^p \right\} dx \\ &\leq C_P \int_{B_\rho(x_0)} \{ |Du - D\ell_{x_0, \rho}|^2 + |Du - D\ell_{x_0, \rho}|^p \} dx \\ &\leq 2C_P \int_{B_\rho(x_0)} |Du - (Du)_{x_0, \rho}|^2 dx + 2C_P |(Du)_{x_0, \rho} - D\ell_{x_0, \rho}|^2 \\ &\quad + 2^{p-1} C_P \int_{B_\rho(x_0)} |Du - (Du)_{x_0, \rho}|^p dx + 2^{p-1} C_P |(Du)_{x_0, \rho} - D\ell_{x_0, \rho}|^p \\ &\leq 2^p C_P^2 \left(\sqrt{n(n+2)} \right)^p \int_{B_\rho(x_0)} \{ |Du - (Du)_{x_0, \rho}|^2 + |Du - (Du)_{x_0, \rho}|^p \} dx. \end{aligned} \quad (4.29)$$

Moreover, for any $\alpha \in (0, 1)$ and $\rho \leq 1$, again using the Poincaré inequality, there holds

$$\begin{aligned} C_\alpha(x_0, \rho) &= \rho^{2\alpha} \int_{B_\rho(x_0)} |u - u_{x_0, \rho}|^2 dx \\ &\leq \rho^{2-2\alpha} \int_{B_\rho(x_0)} |Du|^2 dx \\ &\leq 2 \int_{B_\rho(x_0)} |Du - (Du)_{x_0, \rho}|^2 dx + 2\rho^{2-2\alpha} |(Du)_{x_0, \rho}|^2. \end{aligned} \quad (4.30)$$

Recalling the definitions of Σ_1 and Σ_2 , the estimates (4.29) and (4.30) imply the existence of a radius $0 < \rho < \min\{\rho_*, \text{dist}(x_0, \partial\Omega)\}$ satisfying

$$\Psi(x_0, \rho, \ell_{x_0, \rho}) < \varepsilon_* \quad \text{and} \quad C_\alpha(x_0, \rho) < \kappa_*,$$

for the constants $\rho_*, \varepsilon_*, \kappa_* > 0$ determined in Lemma 4.5. Using the absolute continuity of the integral, there exists a neighbourhood $U \subseteq \Omega$ of x_0 with

$$\Psi(x, \rho, \ell_{x, \rho}) < \varepsilon_* \quad \text{and} \quad C_\alpha(x, \rho) < \kappa_*,$$

for all $x \in U$. Then Lemma 4.5 yields that we have estimates

$$\Psi(x, \theta^k \rho, \ell_{x, \rho}) < \varepsilon_* \quad \text{and} \quad C_\alpha(x, \theta^k \rho) < \kappa_*, \quad (4.31)$$

for all $x \in U$ and $k \in \mathbb{N}$. Here $\theta \in (0, 1/8]$ is independent of the particular point x . This implies

$$\sup_{y \in U, \sigma \in (0, \rho)} \sigma^{-2\alpha} \int_{B_\sigma(y)} |u - u_{y, \sigma}|^2 dx = \sup_{y \in U, \sigma \in (0, \rho)} C_\alpha(y, \sigma) < \theta^{-n-2\alpha} \kappa_* < \infty,$$

and hence $u \in C^{0, \alpha}(U, \mathbb{R}^N)$ by Campanato's characterization of Hölder continuous functions. \square

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